Student Notes

Student Session Topic: Computing Antiderivatives and Definite Integrals

In this prep session we look at the computation of antiderivatives and definite integrals that you may be asked to do on the AP Calculus exams. There are no applications here; they will be considered in several other student study sessions. However, you may be required to compute an antiderivative as part of an application problem.

The antiderivatives that appear on the AP exams are probably a lot simpler than many you have done in class. The exam does not include "monster" integrands. They are testing that you know the basic idea and can apply the various techniques; there are no need for complicated examples to test this.

We will consider mainly multiple-choice questions. Antiderivative on the free-response sections are almost entirely in the context of the problems and not stand alone like those on the multiple-choice. Often they are on the calculator allowed section and there is no need to find the antiderivative.

AB Students should be able to do integrals that come directly from the differentiation of the common functions and those with simple *u*-substitutions including change the limits of integration.

BC students should be able to do all of the above and integration by parts, partial fractions and improper integrals. These notes contain two pages reviewing integration by parts and partial fraction techniques. If you're taking the AB exam you may skip these and some of the multiple-choice questions.

Some hints and notes on the multiple-choice problem in this handout:

Please try the problems first, and then read the notes. When you get home you can see them all being worked at the URL in the box below.

WATCH and LISTEN to the multiple-choice questions being solved

Go to http://tinyurl.com/NMSI-Math-5 Click on the "Full Screen" arrow.
Then click anywhere on the page to see and hear from that point on.

Click anywhere to go back anytime.

Student Notes

For AB and BC students

1. (2003 – AB 5) Very simple, also tests knowing the trig functions of special angles.

2. (2003 – AB 2) A very simple *u*-substitution. Students can shortcut a little here. One just needs to think $u=-4x$ and think $du=-4dx$ then write

$$-\frac{1}{4}\int_0^1 \underbrace{e^{-4x}}_{e^{u}}\underbrace{(-4dx)}_{du}=-\frac{1}{4}e^{-4x}\Big|_0^1=-\frac{1}{4}e^{-4}+\frac{1}{4}e^{0}=-\frac{1}{4e^4}+\frac{1}{4}$$

   This also eliminates the need for changing the limits of integration or back-substituting.

3. (1997 – AB 6) Straightforward

4. (2003 – AB 8) A little more complicated *u*-substitution, but about as hard as they get. (See next problem)

5. (2003 – AB 11) Here the substitution is given and the real emphasis is changing the limits of integration.

6. (1998 – AB 7) Straightforward, divide first.

7. (1998 – AB 20) Easy enough to do by finding the antiderivative, but considering the integral gives the area of some region; the answer is obvious with no computation.

8. (1997 – AB 1) Straightforward

9. (1998 – AB 27) A bit more difficult *u*-substitution.

10. (1997 – BC 89 suitable for AB) A calculator problem. $\int_1^4 \frac{x^2}{1+x^5}dx = f(4)-f(1)$. The definite integral can be found using a calculator and $f(1)$ is given: solve for $f(4)$

11. (1998 – AB 88) A calculator problem. Similar to the previous question. The antiderivative can be found by hand ($\int u^3 du$) but using a calculator her is the smart way.

12. (1998 – BC 82 Suitable for AB) Theory

BC only questions using integration by parts, partial fractions and improper integrals:

13. (2003 – BC 26) A full partial fractions problem with very easy arithmetic. Seethe note on partial fractions below.

14. (1998 – BC 4) Partial fractions with factoring required.

15. (1998 – BC 15) Integration by parts

16. (1997 – BC 84) Integration by parts. This question would now be on the no calculator section since a CAS calculator could do this easily. The TI89 came into general use the following year. See the notes on integration by parts below.

17. (1997 – BC 11) Improper integral

18. (2003 – BC 6) Improper integral.

Multiple-choice answers:

1 d, 2 d, 3 c, 4 b, 5 c, 6 e, 7 a, 8 c, 9 a, 10 d, 11 c, 12 b, 13 d, 14 a, 15 b, 16 c, 17 c, 18 c.

Free-response questions: Free-response questions sometimes require integration by hand. These are usually straightforward and you will see them in the various applications.

<u>The next three pages are notes for BC students on Integration by Parts and Partial Fractions. The AB Calculus exam does not test these techniques.</u>

A note on integration by parts for integrands that require several steps:

Sometimes the integral must be done in several steps. Because of the x^3 in the integrand integration by parts will have to done several times. Here an example of an efficient way to do a longer problem.

Example: Find: $\int x^3 \cos(x)\,dx$

a. Make a table with three columns as shown below.
   - The <u>first column</u> starts with a "+" sign and then alternates signs.
   - In the <u>second column</u> start with *u* and list its derivatives.
   - In the <u>third column</u>, start with *dv* and integrate.
   - Multiply as indicated by the colors and write the solution at the right. (Multiply the first two columns then drop down one row and multiply the product by this expression, write the result on the right. Repeat this for each row.
   - The antiderivative is the sum of the expressions on the right:

Student Notes

$\pm$	u and du	dv and $\int$	
+	x^3	$\cos(x)$	
−	$3x^2$	$\sin(x)$	$+x^3\sin(x)$
+	$6x$	$-\cos(x)$	$-3x^2(-\cos(x))$
−	6	$-\sin(x)$	$6x(-\sin(x))$
+	0	$\cos(x)$	$-6\cos(x)$

- $\int x^3\cos(x)\,dx = x^3\sin(x)+3x^2\cos(x)-6x\sin(x)-6\cos(x)+C$

The quick way to do partial fractions

Example: Write $\dfrac{10x-18}{3x^2-14x-5}$ as the sum of fractions.

a. Factor the denominator and write the sum this way with unknown numerators:

$$\frac{10x-18}{3x^2-14x-5}=\frac{A}{3x+1}+\frac{B}{x-5} \tag{0.1}$$

b. Then multiplying both sides by the common denominator gives.

$$\begin{aligned}10x-18&=A(x-5)+B(3x+1)\\&=(A+3B)x+(-5A+B)\end{aligned} \tag{0.2}$$

c. We can find *A* and *B* by equating the coefficient of *x* and the constant term on each side:

$$\begin{aligned}A+3B&=10\\-5A+B&=-18\end{aligned} \tag{0.3}$$

This system of linear equations can be solved easily enough. **But there is an easier way**.

d. Return to (0.2)

$$10x-18=A(x-5)+B(3x+1) \tag{0.4}$$

Substitute values of *x* that makes one of the factors in parentheses 0. Start with *x* = 5

Student Notes

$$\begin{aligned} 10(5)-18 &= A(5-5)+B(3(5)+1) \\ 32 &= 16B \\ 2 &= B \end{aligned} \tag{0.5}$$

Then let *x* = $-\frac{1}{3}$

$$\begin{aligned} 10\left(-\tfrac{1}{3}\right)-18 &= A\left(-\tfrac{1}{3}-5\right)+B\left(3\left(-\tfrac{1}{3}\right)+1\right) \\ -\tfrac{10}{3}-18 &= -\tfrac{16}{3}A+0 \\ -10-54 &= -16A \\ -64 &= -16A \\ 4 &= A \end{aligned} \tag{0.6}$$

Thus

$$\frac{10x-18}{3x^2-14x-5}=\frac{4}{3x+1}+\frac{2}{x-5} \tag{0.7}$$

Name: ________________________ Class: ___________________ Date: __________ ID: A

Techniques of Integration

Multiple Choice

Identify the choice that best completes the statement or answers the question.

____ 1. $\int_0^{\frac{\pi}{4}} \sin x \, dx =$

a. $-\frac{\sqrt{2}}{2}$

b. $\frac{\sqrt{2}}{2}$

c. $-\frac{\sqrt{2}}{2} - 1$

d. $-\frac{\sqrt{2}}{2} + 1$

e. $\frac{\sqrt{2}}{2} - 1$

____ 2. $\int_0^1 e^{-4x} \, dx =$

a. $\frac{-e^{-4}}{4}$

b. $-4e^{-4}$

c. $e^{-4} - 1$

d. $\frac{1}{4} - \frac{e^{-4}}{4}$

e. $4 - 4e^{-4}$

____ 3. $\frac{1}{2} \int e^{\frac{t}{2}} \, dt =$

a. $e^{-t} + C$

b. $e^{-\frac{t}{2}} + C$

c. $e^{\frac{t}{2}} + C$

d. $2e^{\frac{t}{2}} + C$

e. $e^{t} + C$

Name: ________________________ ID: A

____ 4. $\int x^2 \cos\left(x^3\right) dx =$

a. $-\frac{1}{3}\sin\left(x^3\right) + C$
b. $\frac{1}{3}\sin\left(x^3\right) + C$
c. $-\frac{x^3}{3}\sin\left(x^3\right) + C$
d. $\frac{x^3}{3}\sin\left(x^3\right) + C$
e. $\frac{x^3}{3}\sin\left(\frac{x^4}{4}\right) + C$

____ 5. Using the substitution $u = 2x + 1$, $\int_0^2 \sqrt{2x+1}\, dx$ is equivalent to

a. $\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \sqrt{u}\, du$
b. $\frac{1}{2}\int_0^2 \sqrt{u}\, du$
c. $\frac{1}{2}\int_1^5 \sqrt{u}\, du$
d. $\int_0^2 \sqrt{u}\, du$
e. $\int_1^5 \sqrt{u}\, du$

____ 6. $\int_1^e \left(\frac{x^2 - 1}{x}\right) dx =$

a. $e - \frac{1}{e}$
b. $e^2 - e$
c. $\frac{e^2}{2} - e + \frac{1}{2}$
d. $e^2 - 2$
e. $\frac{e^2}{2} - \frac{3}{2}$

____ 7. What are all values of k for which $\int_{-3}^{k} x^2\,dx = 0$?

a. -3
b. 0
c. 3
d. -3 and
e. 3-3, 0, and 3

____ 8. $\int_{1}^{2} \left(4x^3 - 6x\right) dx =$

a. 2
b. 4
c. 6
d. 36
e. 42

____ 9. What is the average value of $y = x^2\sqrt{x^3+1}$ on the interval [0, 2] ?

a. $\frac{26}{9}$
b. $\frac{52}{9}$
c. $\frac{26}{3}$
d. $\frac{52}{3}$
e. 24

____ 10. If f is the antiderivative of $\frac{x^2}{1+x^5}$ such that $f(1) = 0$, then $f(4) =$

a. -0.012
b. 0
c. 0.016
d. 0.376
e. 0.629

____ 11. Let $F(x)$ be an antiderivative of $\frac{(\ln x)^3}{x}$. If $F(1) = 0$, then $F(9) =$

a. 0.048
b. 0.144
c. 5.827
d. 23.308
e. 1,640.250

____ 12. If $f(x) = g(x) + 7$ for $3 \le x \le 5$, then $\int_3^5 \left[f(x) + g(x) \right] dx =$

a. $2\int_3^5 g(x)\,dx + 7$

b. $2\int_3^5 g(x)\,dx + 14$

c. $2\int_3^5 g(x)\,dx + 28$

d. $\int_3^5 g(x)\,dx + 7$

e. $\int_3^5 g(x)\,dx + 14$

____ 13. $\int \frac{2x}{(x+2)(x+1)}\,dx =$

a. $\ln|x+2| + \ln|x+1| + C$
b. $\ln|x+2| + \ln|x+1| - 3x + C$
c. $-4\ln|x+2| + 2\ln|x+1| + C$
d. $4\ln|x+2| - 2\ln|x+1| + C$
e. $2\ln|x| + \frac{2}{3}x + \frac{1}{2}x^2 + C$

____ 14. $\int \frac{1}{x^2 - 6x + 8}\,dx =$

a. $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$

b. $\frac{1}{2}\ln\left|\frac{x-2}{x-4}\right| + C$

c. $\frac{1}{2}\ln\left|(x-2)(x-4)\right| + C$

d. $\frac{1}{2}\ln\left|(x-4)(x+2)\right| + C$

e. $\ln\left|(x-2)(x-4)\right| + C$

____ 15. $\int x\cos x\,dx =$

a. $x\sin x - \cos x + C$
b. $x\sin x + \cos x + C$
c. $-x\sin x + \cos x + C$
d. $x\sin x + C$
e. $\frac{1}{2}x^2\sin x + C$

____ 16. $\int x^2\sin x\,dx =$

a. $-x^2\cos x - 2x\sin x - 2\cos x + C$
b. $-x^2\cos x + 2x\sin x - 2\cos x + C$
c. $-x^2\cos x + 2x\sin x + 2\cos x + C$
d. $\frac{-x^3}{3}\cos x + C$
e. $2x\cos x + C$

____ 17. $\int_1^{\infty}\frac{x}{\left(1+x^2\right)^2}\,dx$ is

a. $-\frac{1}{2}$
b. $-\frac{1}{4}$
c. $\frac{1}{4}$
d. $\frac{1}{2}$
e. divergent

____ 18. What are all values of p for which $\int_1^{\infty}\frac{1}{x^{2p}}\,dx$ converges?

a. $p < -1$
b. $p > 0$
c. $p > \frac{1}{2}$
d. $p > 1$
e. There are no values of p for which this integral converges.